

Consistent nonadditive approach and nuclear equation of state

A. P. Santos^{1,*}, F. I. M. Pereira^{2,†}, R. Silva^{1,3,‡} and J. S. Alcaniz^{2,§}

¹*Departamento de Física, Universidade Federal do
Rio Grande do Norte, 59072-970 Natal, RN, Brazil*

²*Observatório Nacional, Rua Gal. José Cristino 77,
20921-400 Rio de Janeiro RJ, Brasil and*

³*Universidade do Estado do Rio Grande do Norte, 59610-210, Mossoró, RN, Brasil*

By using a q -calculus, the Walecka many-body field theory was studied in the context of the Tsallis framework. The most important aspect of the application of the q -calculus to the nonadditive formulation of QHD-I is that it naturally emerges as a thermodynamically consistent theory.

PACS numbers:

Keywords: Quantum hadrodynamics; Nuclear Matter

I. INTRODUCTON

The properties of nuclear matter have been investigated through the relativistic phenomenological approach introduced by Walecka *et al.* [1–3]. This framework, also called quantum hadrodynamics (QHD-I), has been an important approach to investigate the behavior of strong interactions at the hadronic energy scales. From the thermodynamical point of view, this model provides a consistent framework for the description of bulk static properties of strong interacting many-body nuclear systems. As a matter of fact, QHD-I is a strong-coupling renormalizable field theory of nucleons interacting via the exchange of scalar (σ) and vector (ω) mesons [1–3]. Although being a consistent framework, some limitations of QHD-I have been investigated considering either a more complete theoretical study for zero

*Electronic address: alysonpaulo@dfte.ufrn.br

†Electronic address: flavio@on.br

‡Electronic address: raimundosilva@dfte.ufrn.br

§Electronic address: alcaniz@on.br

temperature [4, 5] or by introducing the thermal field dynamics in hot nuclear matter [6]. However, we are investigating the nonadditive effects on the QHD-I, as originally developed in [1–3]. QHD-I has been used in the calculations of nuclear matter and finite nuclei (See [7] and references therein), as well as in the astrophysics context to describe the properties of nuclear matter in compact stars [8–14]. In this regard, some improvements and extensions of the model have been made to make it more suitable for neutron and proto-neutron stars calculations [8–14].

On the other hand, the nonadditive statistics has received considerable attention, both from theoretical and observational viewpoints [15]. From the high energy point of view, the very first application was addressed to the solar neutrino problem by the use of the Tsallis framework to derive a distribution function for the interior plasma [16, 17]. Recently, by using the nonadditive statistics, the distribution of transverse momenta in high-energy collisions of proton-proton, and heavy nuclei (e.g., Pb-Pb and Au-Au) have been object of a lot of investigations [18–23]. Actually, the q -distribution has been tested in high energy physics, and results have presented very high quality fits of the transverse momentum distributions made by the STAR [24] and PHENIX [25] collaborations at RHIC and by the ALICE [26] and CMS [27] collaborations at LHC. More recently, an interesting discussion about the Renyi and Tsallis formulas has been investigated through a mathematical formulation in the context of ideal gas [28]. Specifically, Tsallis and Renyi entropies are related as $S_{Tsallis} = K S_{Renyi} = \sum_i p_i K(-\ln p_i)$, with $K(S) = \frac{1}{a}(e^{aS} - 1)$ and $a = 1/C_0$. C_0 is the positive heat capacity and a is a real parameter. This approach has been successfully discussed in the context of black hole thermodynamics and quark-gluon plasma [29, 30].

The connection between the Tsallis statistics and the relativistic nuclear equation of state (EoS) has also been discussed in the context of high energy physics (see, e.g., [31]). In particular, the EoS plays an important role in the determination of the structure and evolution of the proto-neutron stars [32]. In this regard, the calculation of EoS was performed in the context of Tsallis framework and the main physical properties of proto-neutron stars as well as their astrophysical implications have been discussed in [33].

In the attempt to implement the Tsallis statistics on QHD-I, we have proposed a non-additive EoS through a straightforward generalization of Fermi-Dirac distribution [34, 35]. This issue has been claimed, in a recent connection between the nonadditive framework and high energy physics, with respect to the so-called thermodynamical consistency [36–38]. In

this concern, by considering the first and second laws of thermodynamics, the authors have shown that such consistency is possible in the context of Tsallis framework [36–38].

Until recently, the attempts to derive a nonadditive EoS for QHD-I, from first principles of Quantum Statistical Mechanics (QSM), have appeared as an unthinkable task to be achieved. Fortunately, the developments on the so-called q -calculus has opened an important possibility to overcome this difficulty showing that it is a powerful method that has been frequently used to investigate many problems, e.g. connections with Quantum Statistical Mechanics [39, 40].

In this paper, we present a new approach for the thermodynamical consistency of QHD-I. By following the q -calculus [41, 42], we show that the new scheme provides a calculation for the Walecka nuclear EoS based on first principles of the QSM. As a matter of fact, instead of earlier calculations, we have found that thermodynamical quantities describing the EoS, i.e., pressure, energy density and number density depend explicitly on the nonadditive parameter q . The important result of the present work is that the thermodynamical consistency naturally has emerged from first principles without any additional assumption. Moreover, from the mathematical point of view, both the pressure and energy density expressions are equivalent to the ones that have been introduced through other approaches [43–48]. However, it must also be emphasized that, before the development of the q -calculus, prescriptions about the way as the non-additive distribution functions $n_{\mathbf{k}}(T, \nu)$ and $\bar{n}_{\mathbf{k}}(T, \nu)$ enter equations of pressure, energy density, number density and so on, have been considered *a posteriori* by the use of thermodynamical arguments to achieve the claimed consistency [36–38]. Finally, we show that the q -calculus also provides a prescription for the scalar density expression in terms of the powers of the non-additive distribution functions, $n_{\mathbf{k}}^q(T, \nu)$ and $\bar{n}_{\mathbf{k}}^q(T, \nu)$, for positive values of q .

The aim of the present work is twofold. First, our purpose is to use the q -calculus to solve, from first principles, the commented thermodynamical inconsistency in [34, 35]. Second, the simplification adopted here by the use of QHD-I¹ allows us to show, in a more consistent way, that the q -calculus is general to be used in any quantum system.

This paper is organized as follows. In Section II, we present the basic formalism of the

¹ For a more complete theoretical treatment (for zero temperature), see [49]. A discussion on the thermal field dynamics in hot nuclear matter was presented [50].

QHD-I theory as well as its connection with Tsallis framework through the q -calculus. The main results are discussed in Section III. Finally, we summarize our main conclusions in Section IV.

II. WALECKA EQUATION OF STATE IN THE Q-CALCULUS

A. QHD-I: Basic Equations

We now recall the main aspects of QHD-I. The Lagrangian is given by (for details, see [3])

$$\mathcal{L} = \bar{\psi}[(i\gamma_\mu(\partial^\mu - g_\omega\omega^\mu) - (M - g_\sigma\sigma)]\psi + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2) \quad (1)$$

$$- \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu, \quad (2)$$

where $\omega_{\mu\nu}$ is the field tensor

$$\omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu. \quad (3)$$

The Euler-Lagrange equations for the fields and nucleons are given by

$$(\square + m_\sigma^2)\sigma = g_\sigma\bar{\psi}\psi, \quad (4)$$

$$\partial_\mu\omega^{\mu\nu} + m_\omega^2\omega^\nu = g_\omega\bar{\psi}\gamma^\nu\psi, \quad (5)$$

$$[\gamma^\mu(i\partial_\mu - g_\omega\omega_\mu) - (M - g_\sigma\sigma)]\psi = 0, \quad (6)$$

with the scalar source $\bar{\psi}\psi$ and the (conserved) baryon current $B^\nu = \bar{\psi}\gamma^\nu\psi$. In the case of a system with uniform baryon and scalar densities, the meson fields are replaced by the classical fields giving rise to the mean-field equations

$$\sigma = \frac{g_\sigma}{m_\sigma^2}\langle\bar{\psi}\psi\rangle \equiv \frac{g_\sigma}{m_\sigma^2}n_S, \quad (7)$$

$$\omega_0 = \frac{g_\omega}{m_\omega^2}\langle\psi^\dagger\psi\rangle \equiv \frac{g_\omega}{m_\omega^2}n_B \quad (8)$$

$$[\gamma^\mu i\partial_\mu - g_\omega\gamma^0\omega_0 - (M - g_\sigma\sigma)]\psi = 0, \quad (9)$$

with the scalar density $n_S = \langle\bar{\psi}\psi\rangle$ and baryon (vector) density $n_B = \langle\psi^\dagger\psi\rangle$; m_σ and m_ω are the corresponding masses of the scalar and vector meson fields. The third term in Eq. (9) corresponds to the effective mass defined by

$$M^* = M - g_\sigma\sigma. \quad (10)$$

The QHD-I Hamiltonian and baryon number operator are given by [1, 3]

$$\hat{H} = \sum_{\mathbf{k}\lambda} E^*(k) (A_{\mathbf{k}\lambda}^\dagger A_{\mathbf{k}\lambda} + B_{\mathbf{k}\lambda}^\dagger B_{\mathbf{k}\lambda}) + g_\omega \omega_0 \hat{B} - V \left(\frac{1}{2} m_\omega^2 \omega_0^2 - \frac{1}{2} m_\sigma^2 \sigma^2 \right), \quad (11)$$

$$\hat{B} = \sum_{\mathbf{k}\lambda} (A_{\mathbf{k}\lambda}^\dagger A_{\mathbf{k}\lambda} - B_{\mathbf{k}\lambda}^\dagger B_{\mathbf{k}\lambda}), \quad (12)$$

where V stands for the volume of the system and $E^*(k) = \sqrt{k^2 + M^{*2}}$. In order to calculate the scalar density n_S in the context of the q -calculus, we need here an expression for the corresponding operator as done above for \hat{B} . To this end, by following the steps from Eq. (3.28) to Eqs. (3.41) - (3.44) in [3], we obtain the expression for a scalar density operator, which we call \hat{D}_S , given by

$$\begin{aligned} \hat{D}_S &= \frac{1}{V} \sum_{\mathbf{k}\lambda} \frac{M^*}{E^*(k)} (A_{\mathbf{k}\lambda}^\dagger A_{\mathbf{k}\lambda} + B_{\mathbf{k}\lambda}^\dagger B_{\mathbf{k}\lambda}) \\ &= \frac{1}{V} \frac{\partial \hat{H}}{\partial M^*}, \end{aligned} \quad (13)$$

where the last line follows from Eq. (11). From Eqs. (11) and (12) it follows that

$$\begin{aligned} \hat{H} - \mu \hat{B} &= \sum_{\mathbf{k}\lambda} [(E^* - \mu + g_0 \omega_0) A_{\mathbf{k}\lambda}^\dagger A_{\mathbf{k}\lambda} + (E^* + \mu - g_0 \omega_0) B_{\mathbf{k}\lambda}^\dagger B_{\mathbf{k}\lambda}] \\ &\quad - V \left(\frac{1}{2} m_\omega^2 \omega_0^2 - \frac{1}{2} m_\sigma^2 \sigma^2 \right). \end{aligned} \quad (14)$$

In order to shorten the notation, we write $\hat{N}_{\mathbf{k}\lambda} = A_{\mathbf{k}\lambda}^\dagger A_{\mathbf{k}\lambda}$, $\hat{\bar{N}}_{\mathbf{k}\lambda} = B_{\mathbf{k}\lambda}^\dagger B_{\mathbf{k}\lambda}$, $a_{\pm, \mathbf{k}\lambda} = E^*(k) \mp \nu$, where $\nu \equiv \mu - g_\omega \omega_0$ (which plays the role of an *effective chemical potential* as can be seen bellow in Eq. (36)) of the present paper, and $b = V(m_\omega^2 \omega_0^2 - m_\sigma^2 \sigma^2)/2$. So, Eq. (14) is reduced to

$$\hat{H} - \mu \hat{B} = \sum_{\mathbf{k}\lambda} (a_{+, \mathbf{k}\lambda} \hat{N}_{\mathbf{k}\lambda} + a_{-, \mathbf{k}\lambda} \hat{\bar{N}}_{\mathbf{k}\lambda}) - b, \quad (15)$$

with the correspondence $\hat{N}_{\mathbf{k}\lambda}(\hat{\bar{N}}_{\mathbf{k}\lambda}) \rightarrow a_{+, \mathbf{k}\lambda}(a_{-, \mathbf{k}\lambda}) \rightarrow \text{baryon(antibaryon)}$. The partition function (for details, see [51], Chapter 2, pages 36-38), in the Boltzmann-Gibbs statistical mechanics, is given by

$$\begin{aligned} \mathcal{Z}_{BG} &= \sum_{n_1, \dots, \bar{n}_1, \dots} \langle n_1, \dots, \bar{n}_1, \dots | \exp \left[-\beta \sum_i (a_{+, i} \hat{N}_i + a_{-, i} \hat{\bar{N}}_i) + \beta b \right] | n_1, \dots, \bar{n}_1, \dots \rangle \\ &= \sum_{n_1, \dots, \bar{n}_1, \dots} \langle n_1, \dots, \bar{n}_1, \dots | \exp \left[-\beta \sum_i (a_{+, i} n_i + a_{-, i} \bar{n}_i) + \beta b \right] | n_1, \dots, \bar{n}_1, \dots \rangle. \end{aligned} \quad (16)$$

where $n_i(\bar{n}_i)$ are the baryon(antibaryon) occupation numbers in the state $i \equiv \mathbf{k}\lambda$, and $\beta = 1/k_B T$. Notice that $\exp[-\beta \sum_i (a_{+,i} n_i + a_{-,i} \bar{n}_i)]$ is a c-number, which is a classical number (to distinguish it from a quantum observable which is an operator). Now, our goal is to derive the pressure, energy density, and vector and scalar densities within the nonadditive QSM. To this end, by using the q -calculus, we first obtain the nonadditive generalization of Eq. (16).

B. QHD-I and nonadditive framework

In order to introduce the nonadditive effects under the QHD-I, let us consider this new possibility by investigating the issue from first principles. As well known, the temporal evolution of quantum system is given by [51]

$$i\hbar \frac{\partial \Psi(r, t)}{\partial t} = \hat{H} \Psi(r, t). \quad (17)$$

The evolution operator is unitary and transform a state at instant t_0 in another one at t , i.e. $\Psi(r, t) = U(t, t_0) \Psi(r, t_0)$, with $U(t_0, t_0) = \hat{1}$. By using this evolution operator, we can rewrite the expression (17) as

$$\left[i\hbar \frac{\partial U(t, t_0)}{\partial t} - \hat{H} U(t, t_0) \right] \Psi(r, t_0) = 0, \quad (18)$$

and the solution is given by

$$U(t, t_0) = \exp[-i\hat{H}(t - t_0)/\hbar]. \quad (19)$$

Now, we introduce nonadditive effects, by using an evolution operator in the context of Tsallis framework. These effects have considered statistical correlations and non-linearity in the temporal evolution of the system. In this regard, a generalization of expression (19) has been proposed in [52] to be of the form

$$U_q(t, t_0) = \exp_q[-i\hat{H}(t - t_0)/\hbar], \quad (20)$$

where \exp_q is the generalized exponential function. We here adopt the prescription given in [53] by

$$\exp_q(x) := \begin{cases} [1 + (q - 1)x]^{\frac{1}{q-1}} & \text{if } x > 0 \\ [1 + (1 - q)x]^{\frac{1}{1-q}} & \text{if } x \leq 0, \end{cases}$$

which better describes the nonadditive quantum systems (fermions and bosons). By expanding $U_q(t, t_0)$, we obtain

$$U_q(t, t_0) = \hat{1} - \frac{-i}{\hbar} \hat{H} t + \sum_{n=2}^{\infty} \frac{1}{n!} \left[\frac{-i}{\hbar} \hat{H} U(t, t_0) \right]^n \\ \times (1-q)^{n-1} \prod_{k=1}^{n-1} \left(\frac{1}{1-q} - k \right), \quad (21)$$

with $U_q(t_0, t_0) = \hat{1}$. We may describe Eq. (20) in terms of the eigenstates of \hat{H} with $t_0 = 0$ as

$$U_q(t) = \sum_n e_q^{-iE_n t/\hbar} |\Psi_n\rangle \langle \Psi_n|. \quad (22)$$

Introducing the substitution $t = -i\hbar\beta$ (Wick rotation) [54] in the above expression, we obtain

$$U_q(-i\hbar\beta) = \sum_n e_q^{-\beta E_n} |\Psi_n\rangle \langle \Psi_n| = Z_q(\beta). \quad (23)$$

By recognizing β as inverse of $k_B T$, where k_B is the Boltzmann constant and T is the absolute temperature of the system, we have $Z_q(\beta)$ as the generalized partition function of the quantum system. Now, let us define the generalized density operator related to canonical ensemble given by

$$\hat{\rho}_q(\beta) = \frac{e_q^{-\beta \hat{H}}}{Z_q(\beta)}. \quad (24)$$

Similarly, for the grand canonical ensemble, by taking the correspondence $\hat{H} \rightarrow \hat{H} - \mu \hat{B}$, we have the respective density operator

$$\hat{\rho}_q(\beta) = \frac{e_q^{-\beta(\hat{H} - \mu \hat{B})}}{\Xi_q}, \quad (25)$$

where

$$\Xi_q = \text{Tr}[e_q^{-\beta(\hat{H} - \mu \hat{B})}] \\ = \sum_{n_1, n_2, \dots} \langle n_1, n_2, \dots | e_q^{-\beta(\hat{H} - \mu \hat{B})} | n_1, n_2, \dots \rangle \quad (26)$$

is the generalized grand canonical partition function. Let us mention that, until this point, the framework is completely general since one can introduce any physical system through of its Hamiltonian.

At this point, by following the same steps to obtain the Boltzmann-Gibbs partition function in Eq. (16), we use q -calculus [41, 42]. So, from Eqs. (15) and (26), we obtain the generalized QHD-I grand canonical partition function

$$\begin{aligned}\Xi_q &= \sum_{n_1, \dots, \bar{n}_1, \dots} \langle n_1, \dots, \bar{n}_1, \dots | \exp_q[-\beta \sum_i (a_{+,i} \hat{N}_i + a_{-,i} \hat{\bar{N}}_i) + \beta b] | n_1, \dots, \bar{n}_1, \dots \rangle \\ &= \sum_{n_1, \dots, \bar{n}_1, \dots} \langle n_1, \dots, \bar{n}_1, \dots | \exp_q[-\beta \sum_i (a_{+,i} n_i + a_{-,i} \bar{n}_i) + \beta b] | n_1, \dots, \bar{n}_1, \dots \rangle.\end{aligned}\quad (27)$$

Using orthogonality relations and properties of q -product [41, 42]

$$\exp_q \left[\sum_i x_i \right] = \prod_i {}_{\otimes_q} e_q^{x_i} = e_q^{x_1} \otimes_q e_q^{x_2} \otimes_q \dots \quad (28)$$

we can rewrite (27) as

$$\begin{aligned}\Xi_q &= e_q^{\beta b} \otimes_q \left[\sum_{n_1, \dots, \bar{n}_1, \dots} \left(\prod_i {}_{\otimes_q} e_q^{-\beta a_{+,i} n_i} \right) \otimes_q \left(\prod_i {}_{\otimes_q} e_q^{-\beta a_{-,i} \bar{n}_i} \right) \right] \\ &= e_q^{\beta b} \otimes_q \left[\sum_{n_1, \dots, \bar{n}_1, \dots} \left(e_q^{-\beta a_{+,1} n_1} \otimes_q e_q^{-\beta a_{+,2} n_2} \dots \right) \otimes_q \left(e_q^{-\beta a_{-,1} \bar{n}_1} \otimes_q e_q^{-\beta a_{-,2} \bar{n}_2} \dots \right) \right] \\ &= e_q^{\beta b} \otimes_q \left[(1 + e_q^{-\beta a_{+,1}}) \otimes_q (1 + e_q^{-\beta a_{+,2}}) \dots \right] \otimes_q \left[(1 + e_q^{-\beta a_{-,1}}) \otimes_q (1 + e_q^{-\beta a_{-,2}}) \dots \right]\end{aligned}\quad (29)$$

Therefore, we obtain

$$\Xi_q = e_q^{\beta b} \otimes_q \left[\prod_i {}_{\otimes_q} (1 + e_q^{-\beta a_{+,i}}) \otimes_q (1 + e_q^{-\beta a_{-,i}}) \right], \quad (30)$$

or, by considering the original notation, we finally get the desired generalization of the partition function for the QHD-I theory,

$$\Xi_q = e_q^{\beta V(m_\omega^2 \omega_0^2 - m_\sigma^2 \sigma^2)/2} \otimes_q \left[\prod_i {}_{\otimes_q} (1 + e_q^{-\beta(E^* - \nu)}) \otimes_q (1 + e_q^{-\beta(E^* + \nu)}) \right]. \quad (31)$$

By taking the q -log of Ξ_q and introducing the continuous limit, we have

$$\ln_q(\Xi_q) = \frac{V \gamma_N}{(2\pi)^3} \int d^3 k \left\{ \ln_q [1 + e_q^{-\beta(E^* - \nu)}] + \ln_q [1 + e_q^{-\beta(E^* + \nu)}] \right\} + \beta V(m_\omega^2 \omega_0^2 - m_\sigma^2 \sigma^2)/2. \quad (32)$$

Here, γ_N is the multiplicity factor ($\gamma_N = 2$ for pure neutron matter and $\gamma_N = 4$ for nuclear matter). By using the known thermodynamical relations [55], after some algebra, the pressure, energy density and baryon number density are given, respectively, by

$$P = \frac{m_\omega^2}{2} \omega_0^2 - \frac{m_\sigma^2}{2} \sigma^2 + \frac{\gamma_N}{3(2\pi)^3} \int d^3 k \frac{k^2}{E^*(k)} [n_{\mathbf{k}}^q(T, \nu) + \bar{n}_{\mathbf{k}}^q(T, \nu)], \quad (33)$$

$$\epsilon = \frac{m_\omega^2}{2}\omega_0^2 + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\gamma_N}{(2\pi)^3} \int d^3k E^*(k) [n_{\mathbf{k}}^q(T, \nu) + \bar{n}_{\mathbf{k}}^q(T, \nu)], \quad (34)$$

$$n_B = \frac{\gamma_N}{(2\pi)^3} \int d^3k [n_{\mathbf{k}}^q(T, \nu) - \bar{n}_{\mathbf{k}}^q(T, \nu)]. \quad (35)$$

where

$$n_{\mathbf{k}}(T, \nu) = \frac{1}{e_q^{\beta[E^*(k)-\nu]} + 1} \quad \text{and} \quad \bar{n}_{\mathbf{k}}(T, \nu) = \frac{1}{e_q^{\beta[E^*(k)+\nu]} + 1}, \quad (36)$$

are the nonadditive distributions for baryons and anti-baryons. Note that, in the $q \rightarrow 1$ limit, the standard Fermi-Dirac distributions $n_{FD}(T, \nu)$ and $\bar{n}_{FD}(T, \nu)$ are recovered. Additionally, as physically expected, when $T \rightarrow 0$ we obtain $n_{\mathbf{k}}(T, \nu) \rightarrow n_{FD}(T, \nu)$ and $\bar{n}_{\mathbf{k}}(T, \nu) \rightarrow \bar{n}_{FD}(T, \nu)$. This amounts to saying that for studies of nuclear matter at zero temperature or neutron stars interiors (where, in nuclear scale, $T \simeq 0$) we do not expect any nonadditive signature. On the other hand, in heavy ions collision experiments or in the interiors of protoneutron stars, with typical stellar temperatures of several tens of MeV (1 MeV = 1.1065×10^{10} K), nonadditive effects can be important.

To calculate the scalar density given by $n_S = \langle \hat{D}_S \rangle$, we assume that the nonadditive expectation value of the scalar density operator can be given by

$$\begin{aligned} \langle \hat{D}_S \rangle &= \frac{\text{Tr}[\hat{D}_S e_q^{-\beta(\hat{H}-\mu\hat{B})}]}{\Xi_q}, \\ &= \frac{1}{V} \frac{1}{\Xi_q} \frac{\partial}{\partial M^*} \text{Tr}[\hat{H} e_q^{-\beta(\hat{H}-\mu\hat{B})}] \end{aligned} \quad (37)$$

and, from Eq. (13), after some algebra, we obtain

$$n_S = \frac{\gamma_N}{(2\pi)^3} \int d^3k \frac{M^*}{E^*(k)} [n_{\mathbf{k}}^q(T, \nu) + \bar{n}_{\mathbf{k}}^q(T, \nu)]. \quad (38)$$

Notice that here n_S is given in terms of $n_{\mathbf{k}}^q(T, \nu)$ and $\bar{n}_{\mathbf{k}}^q(T, \nu)$. We here emphasize that this expression of n_S has also emerged as a consequence of the q -calculus.

Finally, from (7), (10) and (38), we obtain the equation

$$M^* = M - \left(\frac{g_\sigma}{m_\sigma}\right)^2 \frac{\gamma_N}{(2\pi)^3} \int d^3k \frac{M^*}{E^*(k)} [n_{\mathbf{k}}^q(T, \nu) + \bar{n}_{\mathbf{k}}^q(T, \nu)], \quad (39)$$

from which the effective mass M^* is determined self-consistently.

Additionally, we also use for the coupling constants the values given in reference [1–3], namely²,

² For the purpose of the present work, the values given in Eq. (40) suffices to investigate the effects of the nonadditivity in neutron and nuclear matter. Variations of the coupling constants, within the acceptable values given in current literature, do not qualitatively affect the conclusions.

$$\left(\frac{g_\sigma}{m_\sigma}\right)^2 = 11.798 \text{ fm}^2 \quad \text{and} \quad \left(\frac{g_\omega}{m_\omega}\right)^2 = 8.653 \text{ fm}^2, \quad (40)$$

which are fixed to give (at $T = 0$) the bind energy $E_{\text{bind}} = -15.75 \text{ MeV}$ and $k_F = 1.42 \text{ fm}^{-1}$. Other important nuclear quantities such as, for example, the symmetry energy coefficient and compressibility, at saturation density, are calculated at zero temperature (for a review, see [56]). As commented above, at $T = 0$ the effects of nonadditivity do not appear in these quantities. In particular, the problem of the high value of the compressibility predicted by QHD-I, with respect to the empirical one (as shown in Table II of Ref. [3]), is due to the limitations of the theory. Consequently, in the present work, the predictions of QHD-I at $T = 0$ remain the same as in its original form, independently of the value of the q parameter.

III. RESULTS

As done in [34, 35], in the present paper, we have calculated the important quantities that govern the behaviors of pure neutron matter ($\gamma = 2$) and nuclear matter ($\gamma = 4$) for several values of T and q . Generally speaking, the results are qualitatively similar to those in [34, 35], except for some cases which we comment here. More specifically, with respect to the cited previous works, the main aspect of the calculation by using the powers of the non-additive distribution functions, $n_{\mathbf{k}}^q(T, \nu)$ and $\bar{n}_{\mathbf{k}}^q(T, \nu)$, instead of $n_{\mathbf{k}}(T, \nu)$ and $\bar{n}_{\mathbf{k}}(T, \nu)$, is that the numerical results (the curves) for the scalar and vector meson fields, pressure, energy density, and so on, become closer to the corresponding ones for $q = 1$. As an example, we show in Fig. 1, for pure neutron matter, the effective nucleon mass (in panel (a)) and the vector meson field (in panel (b)) as functions of T , at the given values of ν and q . It is evident the similarity of the present results with respect to the ones in Refs. [34, 35]. So, we here do not show all the corresponding figures, except for a few cases which present some differences with respect to the ones in [34, 35].

A. The phase structure at non-zero baryon density

Of particular interest is the behavior of the EoS of the nuclear matter ($\gamma = 4$) around the critical point, where the liquid-gas phase transition changes from first to second order. This phase transition is of the Van der Waals type. As is well known, at temperatures

lower than the critical temperature T_c , the isothermals of the $P \times \epsilon$ plot present three characteristics regions (for more detailed descriptions, see [55] and [57]). The first one, at low energy densities, corresponds to a gas phase of (evaporated) nucleons. The second one, at intermediate energy densities, is a mixed phase made of coexisting nucleon gas and nucleon liquid (formed by clumps or droplets of nucleons). Finally, the third region, at high energy densities, corresponds to the pure nucleon liquid. In the mixed phase, the $P \times \epsilon$ plot oscillates between a maximum and a minimum (with a pronounced dip, in some cases), being usually treated via Maxwell construction. At $T < T_c$, the liquid-gas phase transitions are of first order. As T increases, the intermediate region of the mixed phase shrinks, vanishing at $T = T_c$. In this case, the $P \times \epsilon$ plot has a turning point in which the system passes (along the isotherm) from the gas phase to the liquid phase, characterizing a second order phase transition. For $T > T_c$, the system is in the gas phase [57]. In nonadditive statistical mechanics, these features are dependent not only on the values of the coupling constants of the QHD-I theory, but also on the value of the q parameter. Moreover, the EoS becomes stiffer for increasing values of q .

The dip observed for $q = 1$, that characterizes the first order phase transition at temperatures lower than the critical one, can be reduced if we take values of $q > 1$, suggesting that Maxwell construction can be canceled by properly choosing the values of q . However, this is possible only for a restricted range of T and q . A detailed analysis to obtain the critical temperatures and corresponding nonadditive parameters have shown that for $q = 1$, the critical temperature is $T_c = 20.16$ MeV, illustrated by the lower solid curve in panel (a) of Fig. 2. Below this temperature, Maxwell construction is not eliminated. The upper curve corresponds to $T_c = 20.65$ MeV and $q_c = 1.3$ (which is practically in the convergence limit of the integrals in Eqs. (33)-(35) and (38)). In this range of temperatures, we have calculated several values of T_c and q_c , plotted in panel (b). We remark that the turning points of the isotherms in panel (a) are slightly shifted to right, corresponding to different (increasing) values of the energy density (or the baryon number density). Similarly, in panel (b), the baryon density increases along the curve from left to right. The nonlinearity of the points clearly show the influence of the q power in $n_{\mathbf{k}}^q(T, \nu)$ and $\bar{n}_{\mathbf{k}}^q(T, \nu)$, when compared with the (inconsistent) use of $n_{\mathbf{k}}(T, \nu)$ and $\bar{n}_{\mathbf{k}}(T, \nu)$ in [34, 35].

B. The phase structure at zero baryon density

By considering the same arguments of [58] and [35], we have explored the nonadditive phase structure of QHD-I at vanishing baryon density, for the value of the coupling constant $C_S^2 = 365$, where $C_S^2 = (g_\sigma/m_\sigma)^2 M^2$ [58]. At $n_B = \nu = 0$, the terms with the baryon density do not appear in Eqs. (33) and (34).

In Fig. 3, the panel (a) shows for $q = 1$ a sudden drop in M^* . In reality, the detailed curve, in panel (b), shows that around $T \sim 185$ MeV the self consistent solution of Eq. (39) is triple valued. The pressure and energy density are also triple valued, with abrupt rises at the same temperature, characterizing the first order phase transition. As a consequence of the sudden drop making the effective mass very small at high temperatures ($T \gtrsim 185$ MeV), the system decouples to a nearly free-massless nucleon gas. Here, this depends on the values of C_s^2 and q . As pointed out in [58], this is analogous to the expected chiral phase transition in high-temperature QCD. However, in Walecka theory, there is no liberation of the internal constituents of the nucleons. Thus, the phase transition in QHD theory can not be interpreted as a baryon-quark matter one.

On the other hand, interesting features are obtained for the specific heat. Differently from the treatment discussed in Ref. [58], the mathematical structure of the self-consistency equation in our approach is not simple, so that the calculation must also be done numerically. We observe that the specific heat calculated from Eq. (34) is linear in dM^*/dT . By writing

$$C_H = \frac{d\varepsilon}{dT} = \frac{d\varepsilon}{dM^*} \frac{dM^*}{dT}, \quad (41)$$

we can see from Eq. (39) that

$$\begin{aligned} \frac{dM^*}{dT} &= \frac{-2C_{M^*} \frac{M^*}{T} \int_0^\infty \frac{2k^2 + M^{*2}}{E^*(k)} n_{\mathbf{k}}^q(T, \nu) dk}{1 + 2 C_{M^*} \int_0^\infty \frac{k^2 - M^{*2}}{E^*(k)} n_{\mathbf{k}}^q(T, \nu) dk} \\ &= \frac{-2C_{M^*} \frac{M^{*2}}{T} \int_0^\infty \frac{2k^2 + M^{*2}}{E^*(k)} n_{\mathbf{k}}^q(T, \nu) dk}{M - 2 C_{M^*} M^{*3} \int_0^\infty n_{\mathbf{k}}^q(T, \nu) \frac{dk}{E^*(k)}}, \end{aligned} \quad (42)$$

where

$$C_{M^*} = (g_\sigma/m_\sigma)^2 \gamma_N / \pi^2 \equiv C_S^2 / M^2.$$

The singularities of dM^*/dT lie on the curve determined by the vanishing of the denominator. We observe that, when there is a sudden fall in M^* , a peak arises in the specific

heat around $T \sim 185$ MeV, as shown in panel (a) of Fig. 4. Panel (b) shows the complicated structure of the specific heat (with a negative part) characterizing a first order phase transition. For $q = 1.02$, C_H is single valued and presents a smooth behavior with a peak around $T \sim 180$ MeV (instead of, say, $T \sim 175$ MeV in [35]). We also observe that, at zero baryon density, Eq. (35) gives an equal number of nucleons and anti-nucleons, so its ratio is $N/\bar{N} = 1$ at nonzero temperatures, independently of the values of the q parameter.

IV. FINAL REMARKS

The thermodynamics consistency from the nonadditive framework in high energy physics is fundamental in the construction of a coherent approach. In this paper, we have explored the mean field theory of Walecka (QHD) [1–3, 7] in the context of the non-additive framework without additional requirements. The treatment discussed here differs of the previous results investigated in [34, 35], i.e., a simple substitution of Fermi-Dirac distribution by a nonadditive quantum distribution was considered to calculate the Walecka nuclear EoS.

By considering the so-called q -calculus [41, 42], we have introduced a generalized thermodynamical formulation based only on the first principles of quantum statistical mechanics. The thermodynamics variables describing the nuclear matter, i.e, pressure, energy density, number density and scalar density depend on the nonadditive parameter being given by expressions (33)-(35), (38) and (39).

We have investigated the phase structure of nuclear matter at high temperature, by considering both zero and non-zero baryon density, as well as a pure neutron matter ($\gamma = 2$) and a nuclear matter ($\gamma = 4$). As discussed earlier, differently of previous results based on the inconsistent thermodynamical description [34, 35], we have shown that the nonlinearity from expressions (33)-(35) has not eliminated the Maxwell constructions (except for a very restricted range of temperature). By following similar arguments of [35, 58], we have calculated numerically the specific heat, being that the order of phase transitions also depends of nonadditive parameter q .

The most important aspect of the present work rests on the q -calculus with which we have shown that, from first principles, a thermodynamically consistent calculation of the QHD-I equation of state is possible. In spite of limitations presented by QHD-I, we have used it as an example to show the effects of nonadditivity on the nuclear EoS. We here

have also shown, for the first time, the importance of the q -calculus, due to its generality for application in quantum systems. Mathematically speaking, the present use of q -calculus serve to show that its generalization to extended forms of QHD-I is straightforward.

Although we have applied the q -calculus on the QHD-I theory, the formalism is in principle general for any physical system described by a partition function like that in Eq. (26).

Acknowledgements

The authors (APS, RS and JSA) thank INCT-INEspaço, MCT, CNPq and FAPERJ for the grants under which this work was carried out.

-
- [1] J. D. Walecka, Ann. Phys. **83**, 491 (1974).
 - [2] S. A. Chin , J. D. Walecka, Phys. Lett. B **52**, 24 (1974).
 - [3] B. D. Serot, J.D. Walecka, Advances in Nuclear Physics 16, Plenum Press, 1986.
 - [4] S. A. Chin, Ann. Phys. (N.Y.) **108**, 301 (1977).
 - [5] B. L. Friman and P. A. Henning, Phys. Lett. B **206**, 579 (1988).
 - [6] P. A. Henning, Phys. Rep. **253**, 235 (1995).
 - [7] B. D. Serot, J. D. Walecka, Int. Journ. Mod. Phys. E **6**, 515 (1997).
 - [8] N. K. Glendenning, *Compact Stars, Nuclear Physics, Particle Physics, and General Relativity*, Springer, New York, 2000.
 - [9] D.P. Menezes, C. Providencia, Phys. Rev. C **68**, 035804 (2003).
 - [10] D.P. Menezes, C. Providencia, Braz. J. Phys. **34**, 724 (2004).
 - [11] A.M. Santos, D.P. Menezes, Braz. J. Phys. **34**, 833 (2004).
 - [12] C. Grohmann, et al., Int. J. Mod. Phys. E **16**, 2838 (2007).
 - [13] V.A. Dexheimer, et al., Phys. Rev. C **77**, 065803 (2008).
 - [14] R. Cavagnoli, D.P. Menezes, J. Phys. G: Nucl. Part. Phys. **35**, 115202 (2008).
 - [15] M. Gell-Mann and C. Tsallis, eds. "nonextensive Entropy: Interdisciplinary Applications" (Oxford University Press, New York, 2004).
 - [16] G. Kaniadakis, A. Lavagno, and P. Quariti, Phys. Lett. B **369**,308 (1996).
 - [17] P. Quarati et al., Nucl. Phys. A **621**, 345c (1997).

- [18] G. Wilk and Z. Wlodarczyk, Eur. Phys. J. A **40**, 299 (2009).
- [19] T.S. Biro, G. Purcsel and K. Urmossy, Eur. Phys. J. A **40**, 325 (2009).
- [20] V. Khachatryan et al (CMS Collaboration), J. High Energy Phys. **02**, 041 (2010).
- [21] V. Khachatryan et al (CMS Collaboration), Phys. Rev. Lett. **105**, 022002 (2010).
- [22] M. Shao, L. Yi, Z.B. Tang, H.F. Chen, C. Li and Z.B. Xu, J. Phys. G **37**(8), 085104 (2010).
- [23] T. Wibig, J. Phys. G: Nucl. Part. Phys. **37**, 115009 (2010).
- [24] B. I. Abelev, et al. (STAR Collaboration), Phys. Rev. C **75**, 064901 (2007).
- [25] A. Adare et al. (PHENIX Collaboration), Phys. Rev. C **83**, 064903 (2011).
- [26] K. Aamodt, et al. (ALICE Collaboration), Eur. Phys. J. C **71**, 1655 (2011).
- [27] V. Khachatryan, et al. (CMS Collaboration), J. High Energy Phys. **05**, 064 (2011).
- [28] T. S. Biro, Physica A **392**, 3132 (2013).
- [29] T. S. Biro and V. G. Czinner, Phys. Lett. B **726**, 861 (2013).
- [30] T. S. Biro, G. G. Barnafoldi and P. Van, Eur. Phys. J. A **49**, 110 (2013).
- [31] A. Drago, A. Lavagno and P. Quarati, Physica A **344**, 472 (2004).
- [32] A. Drago, A. Lavagno, I. Parenti, Astrophys. J. **659**, 1519 (2007).
- [33] A. Lavagno and D. Pigato, Eur. Phys. A **47**, 52 (2011)
- [34] F.I.M. Pereira, R. Silva, J.S. Alcaniz, Phys. Rev. C **76**, 015201 (2007).
- [35] F.I.M. Pereira, R. Silva, J.S. Alcaniz, Phys. Lett. A **373**, 4214 (2009).
- [36] J. M. Conroy, H. C. Miller and A. R. Plastino, Phys. Lett. A **374**, 4581 (2010).
- [37] J. Cleymans and D. Worku, Eur. Phys. J. A **48**, 160 (2012).
- [38] J. Cleymans and D. Worku, J. Phys. G: Nucl. Part. Phys. **39**, 025006 (2012).
- [39] R. Silva, D. H. L. Anselmo and J. S. Alcaniz, EPL **89**, 10004 (2010).
- [40] F. D. Nobre, M. A. Rego-Monteiro and C. Tsallis, Phys. Rev. Lett. **106**, 140601 (2011).
- [41] E.P. Borges, Physica A **340**, 95 (2004).
- [42] L. Nivanen, A. Le Mehaute, Q. A. Wang, Rep. Math. Phys. **52**, 437 (2003).
- [43] A. Lavagno, P. Quarati and A. M. Scarfone, Braz. Joun. Phys. **39**, 457 (2009).
- [44] W. M. Alberico and A. Lavagno, Eur. Phys. J. A **40**, 313 (2009).
- [45] A. Lavagno, D. Pigato and P. Quarati, J. Phys. G: Nucl. Part. Phys. **37**, 115102 (2010).
- [46] A. Lavagno and D. Pigato, J. Phys. G: Nucl. Part. Phys. **39**, 125106 (2012).
- [47] G. Gervino, A. Lavagno and D. Pigato, Cent. Eur. J. Phys. **10**, 594 (2012).
- [48] A. Lavagno and D. Pigato, Physica A **392**, 5164 (2013).

- [49] B. L. Friman and P. A. Henning, Phys. Lett. B **206**, 579 (1988).
- [50] P. A. Henning, Phys. Rep. **253**, 235 (1995).
- [51] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems*, McGraw-Hill, New York, 1965.
- [52] U. Tirnakli, S. F. Özeren, F. Büyükkiliç, D. Demirhan, Z. Phys. B **104**, 341 (1997).
- [53] A. M. Teweldeberhan, A. R. Plastino, and H. C. Miller Phys. Lett. A **343**, 71 (2005).
- [54] A. Atland, B. Simons, *Condensed Matter Fields Theory*, Cambridge University Press, Second Edition, (2010).
- [55] J. I. Kapusta, *Finite-Temperature Field Theory*, Cambridge University Press, Cambridge, 1993.
- [56] J. P. Blaizot, Phys. Rep. **64**, 171 (1980).
- [57] J. Pastor, J. C. Caillon and J. Labarsouque, Nucl. Phys. A **64**, 125 (2004).
- [58] Theis, J., Stöcker, H. and Polonyi, J., Phys. Rev. D **28**, 2286(1983).

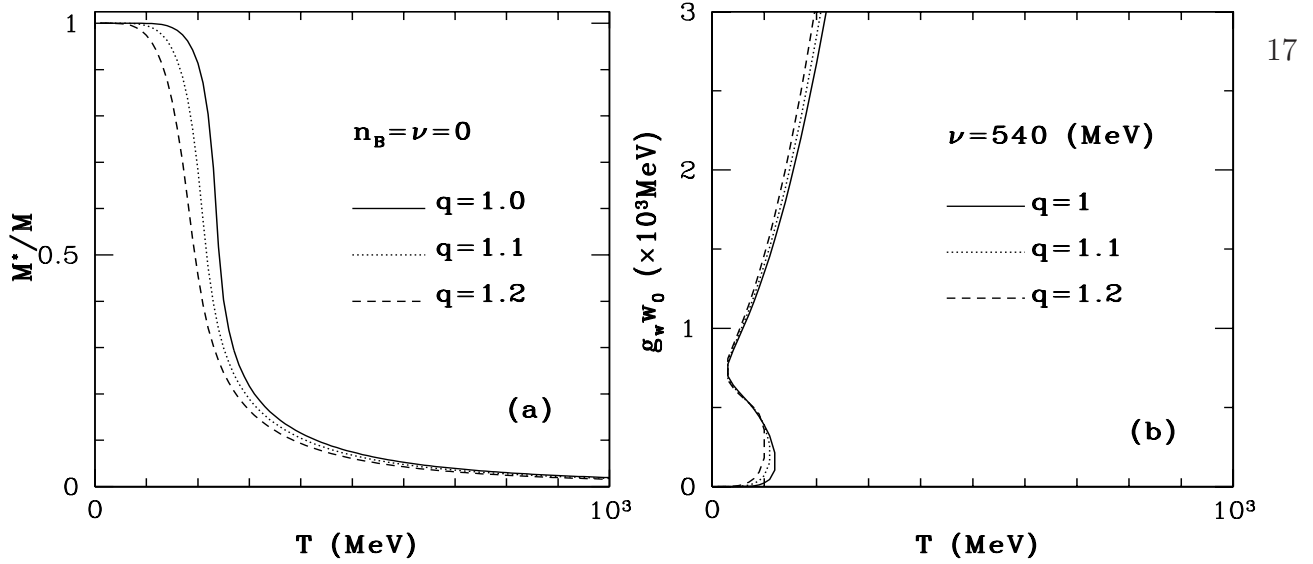


FIG. 1: For pure neutron matter ($\gamma_N = 2$): (a) the self-consistent nucleon mass at zero baryon density as function of temperature for different values of the parameter q . (b) As in (a), but for the vector meson field at nonzero baryon density for the given value of ν .

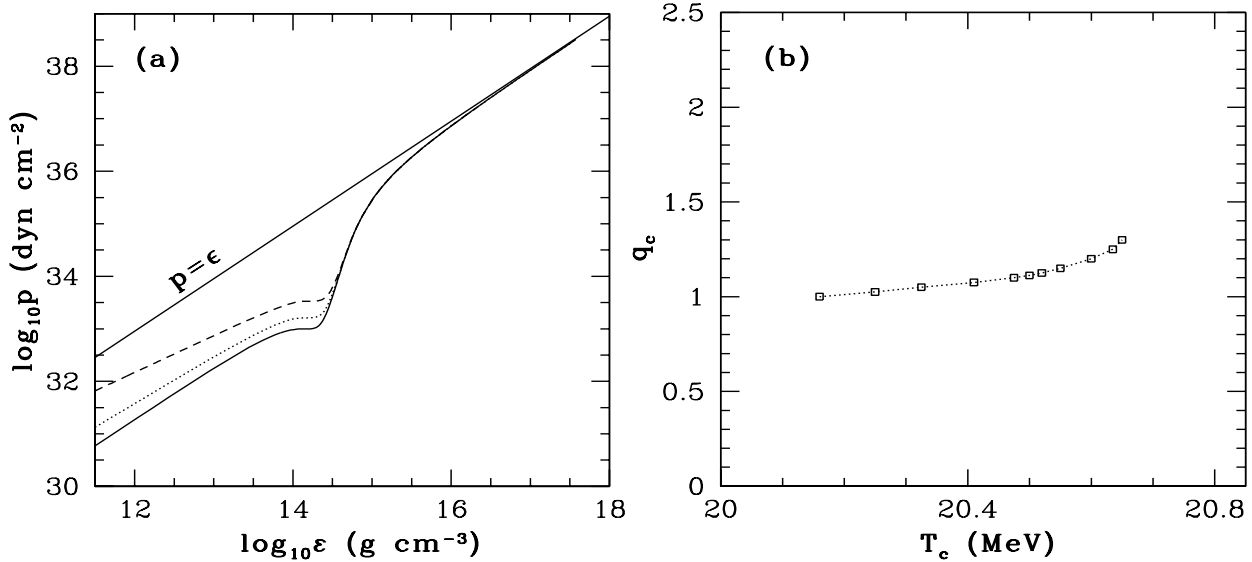


FIG. 2: For nuclear matter ($\gamma_N = 4$): (a) isotherms for three arbitrarily chosen values of the critical temperature T_c and the corresponding parameter q_c for $T_c = 20.15 \text{ MeV}$ and $q_c = 1$ (solid); $T_c = 20.475 \text{ MeV}$ and $q_c = 1.1$ (dots); $T_c = 20.635 \text{ MeV}$ and $q_c = 1.25$ (dash). (b) The critical parameter q_c as function of the critical temperature.

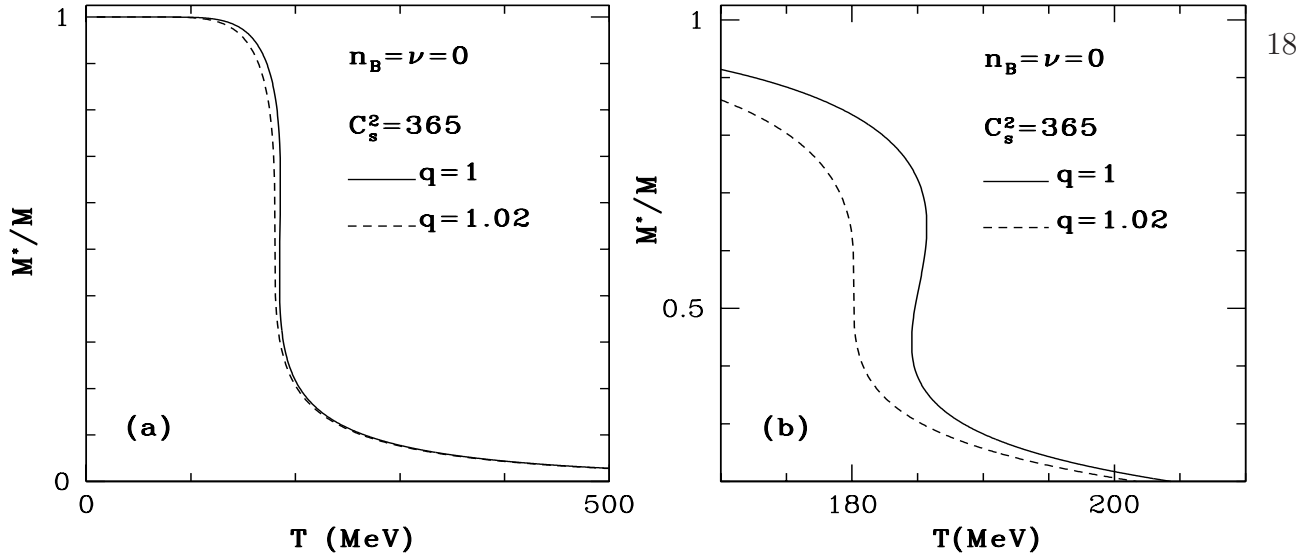


FIG. 3: For nuclear matter ($\gamma_N = 4$): (a) the self-consistent nucleon mass at zero baryon density as function of temperature at the given values of C_s^2 and q . (b) As in (a), but with a stretched temperature region around the point of phase transition.

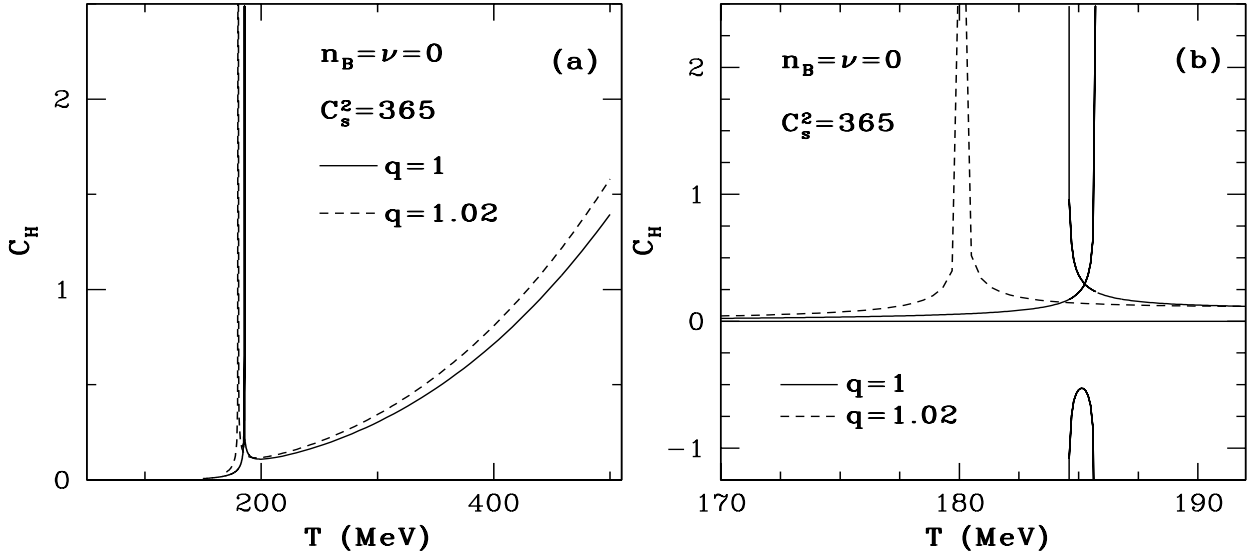


FIG. 4: (a) The specific heat of nuclear matter ($\gamma = 4$) divided by the corresponding $q = 1$ Stefan-Boltzmann limit ($C_H^{SB} = \gamma_N(7\pi^2/30)k_B^4 T^3$) as function of temperature for the given value of C_s^2 and two different values of the parameter q . (b) The same as in (a), but in the stretched region around the phase transition point. For $q = 1$ and $q = 1.02$ the phase transitions are, respectively, of the first and second order.